Compound contribution to the Treasury from the inefficiencies of the pools

Antonio Caccese
csp@capitalstakepool.info

Federico Pasini
stakecool.io@gmail.com

Piero Zanetti
lordwotton29@pm.me

November 2, 2020

Abstract

Shortly before the Shelley hard-fork, changes were introduced in the monetary expansion mechanism to ensure a greater share of rewards for the pools. We have introduced a measure of the inefficiency of pools in achieving full rewards and have shown that this results in a compound contribution to treasury. We propose a mechanism to separate this compound contribution from the calculation of rewards, thus making the share of contributions to the treasury more predictable and transparent.

Definitions

We introduce the following quantities

- T_k = total supply in epoch k.
- ΔT_k = reserves in epoch k.
- R_k = pool rewards in epoch k.
- F_k = fees cumulated in epoch k.
- TR_k = treasury in epoch k.
- η_k = network performance in epoch k.

Reserves and monetary expansion with the latest changes

The basic formula for reserves dynamic from one epoch to the next is

$$\Delta T_{k+1} = (1 - \min\{\eta_k, 1\} \cdot \rho) \cdot \Delta T_k, \tag{1}$$

where ρ is the money expansion rate, that's currently set to 0,3%. However, before the Shelley "hard-fork", it was decided to give back to the reserves the fraction of the pool rewards that should go to the treasury because of the various *inefficiencies*. We recall that the pool rewards in the epoch k are given by the formula

$$R_k = (1 - \tau) \cdot (F_k + \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k), \tag{2}$$

where τ is the treasury taxation rate parameter, that's currently set to 20%.

Let's call i_k the fraction of the pool rewards that goes back to the reserves, $0 \leq i_k \leq 1$. We can see i_k as a *global inefficiency estimator*. This can be disaggregated in several terms, e.g. a pledge part, unstaked ada part etc. With this modified mechanism, the reserves dynamics changes to

$$\Delta T_{k+1} = (1 - \min\{\eta_k, 1\} \cdot \rho) \cdot \Delta T_k + i_k \cdot R_k, \tag{3}$$

so that

$$\begin{aligned} i_k &= \frac{\Delta T_{k+1} - (1 - \min\{\eta_k, 1\} \cdot \rho) \cdot \Delta T_k}{R_k} \\ &= \frac{\Delta T_{k+1} - (1 - \min\{\eta_k, 1\} \cdot \rho) \cdot \Delta T_k}{(1 - \tau) \cdot (F_k + \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k)}. \end{aligned} \tag{4}$$

$$
\begin{tikzcd}
\Delta T_k \arrow[d, "\text{Reward pot}"] \\
F_k + \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k \arrow[d, "\text{Pool rewards}"] \\
R_k \arrow[d, "\text{Inefficieny calculation}"] \\
i_k \cdot R_k \arrow[d, "\text{New reserves calculation}"] \\
\{1 - [1 - i_k \cdot (1-\tau)] \cdot \min\{\eta_k, 1\} \cdot \rho\} \cdot \Delta T_k + i_k \cdot (1-\tau) \cdot F_k
\end{tikzcd}
$$

Figure 1: How monetary expansion flow from reserves in epoch k to new reserves in epoch $k+1$ currently.

The formula 3 can also be written explicitly in terms of the reserves

$$\Delta T_{k+1} = \{1 - [1 - i_k \cdot (1-\tau)] \cdot \min\{\eta_k, 1\} \cdot \rho\} \cdot \Delta T_k + i_k \cdot (1-\tau) \cdot F_k.$$

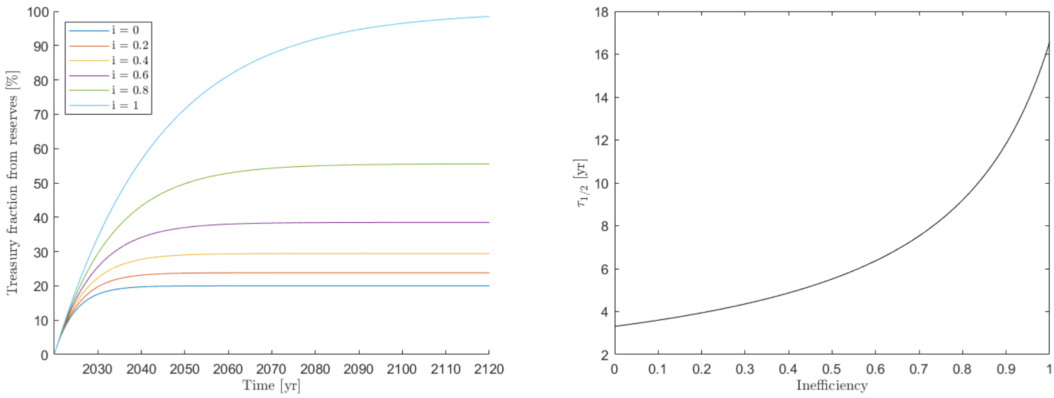
Using the rewards splitting formula from [1] (page 37), we can express the inefficiency as

$$i_k \cdot R_k = R_k - \sum_{i=1}^{N_k} \hat{f}(s_i^{(k)}, \sigma_i^{(k)}, \bar{p}_i^{(k)}) \implies i_k = 1 - \sum_{i=1}^{N_k} \frac{\hat{f}(s_i^{(k)}, \sigma_i^{(k)}, \bar{p}_i^{(k)})}{R_k} \quad (5)$$

where N_k is the number of active pools in epoch k. In the next epoch we have

$$\begin{aligned} R_{k+1} &= (1-\tau) \cdot (F_{k+1} + \min\{\eta_{k+1}, 1\} \cdot \rho \cdot \Delta T_{k+1}) \\ &= (1-\tau) \cdot [F_{k+1} + \min\{\eta_{k+1}, 1\} \cdot \rho \cdot (1 - \min\{\eta_k, 1\} \cdot \rho) \cdot \Delta T_k + i_k \cdot R_k], \end{aligned}$$

where we used 3 in the last passage. Note that R_k is taxed again because of the inefficiency of the previous epoch, so that inefficiency implies a compound taxation (see fig. 1).



(a) (b)

Figure 2: (a) Asymptotic behaviour of the expected fraction of the reserves that went to the treasury. From current data we have estimated $i = 0,61$, so that the actual share of the reward pot that goes to the treasury is around 40% (violet curve). (b) Effective half-life ($\tau_{1/2} = \ln 2 \cdot \hat{\tau}$) of the reserves as a function of the inefficiency i and network performance η. This is the time in years necessary to halve the reserves. $\eta = 0.956$ was estimated from actual network performance data $(0,956 \pm 0,035)$.

Treasury growth

The formula for treasury growth is straightforward

$$TR_{n+1} = TR_n + \frac{\tau}{1-\tau} \cdot R_n = TR_0 + \frac{\tau}{1-\tau} \cdot \sum_{k=0}^{n} R_k. \tag{6}$$

The rewards terms R_k are unpredictable by design because of the unpredictable nature of the fee part of the reward pot, moreover with the latest changes in the rewards distribution there's also an unpredictable term coming from the inefficiency. However, we can neglect the fee part of the reward pot as a first approximation because it's currently negligible, given the current usage of the network, so that the equation 6 can be written

$$TR_{n+1} = TR_0 + \tau \cdot \sum_{k=0}^{n} \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k$$

For the purpose of making an analysis of future trends, we define $\tilde{\rho}_k = [1 - i_k \cdot (1-\tau)] \cdot \rho$ the *effective monetary expansion rate* and we assume

$i_k \approx i$, $\eta \approx \eta < 1$. This approximation implies $\tilde{\rho}_k \approx \tilde{\rho} = [1 - i \cdot (1 - \tau)] \cdot \rho$. Now we can write the reserves at epoch n as

$$\Delta T_n = e^{-n/\hat{\tau}} \cdot \Delta T_0, \tag{7}$$

where we introduced the effective time constant

$$\hat{\tau}(\eta, i, \tau) = -\{\log [1 - \tilde{\rho}(i, \tau) \cdot \eta]\}^{-1},$$

that should not be confused with the treasury rate growth τ. Moreover given the equation 7 and the above approximations, we can write an explicit formula for the treasury at epoch n

$$TR_n = TR_0 + \tau \cdot \eta \cdot \rho \cdot \sum_{k=0}^{n-1} e^{-k/\hat{\tau}} \cdot \Delta T_0 = TR_0 + \frac{\tau \cdot (1 - e^{-n/\hat{\tau}})}{1 - i \cdot (1 - \tau)} \cdot \Delta T_0 \tag{8}$$

from which we can calculate the expected fraction of the reserves that went to the treasury at epoch n (see fig. 2 (a))

$$\frac{TR_n - TR_0}{\Delta T_0} = \frac{\tau \cdot (1 - e^{-n/\hat{\tau}})}{1 - i \cdot (1 - \tau)}.$$

However, it should be noted that inefficiencies lengthen the effective time constant of monetary expansion, thus giving the network more time to reach better performance (see fig. 2 (b)).

$$\begin{tikzcd}
\Delta T_k \arrow[d, "\text{Reward pot}"] \\
F_k + \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k \arrow[d, "\text{Inefficieny calculation}"] \\
i_k \cdot (F_k + \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k) \arrow[d, "\text{Actual pool rewards}"] \\
(1-\tau) \cdot (1-i_k) \cdot (F_k + \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k) \arrow[d, "\text{New reserves calculation}"] \\
[1-(1-i_k) \cdot \min\{\eta_k, 1\} \cdot \rho] \cdot \Delta T_k + i_k \cdot F_k
\end{tikzcd}$$

Figure 3: How monetary expansion flow from reserves in epoch k to new reserves in epoch $k+1$ with CIP mechanism.

CIP: Treasury fraction on actual distributed rewards

In order to make more transparent the flow of the excess expansion that goes back to the reserves, we proposed a CIP in which the reserves contribution from inefficiency is decoupled from the treasury (see fig. 3). In this new scheme, inefficiency is calculated directly on the rewards pot, before the taxation of the treasury occurs. We name the reward pot $RP = F + \min\{\eta, 1\} \cdot \rho \cdot \Delta T$ and the reward splitting formula changes to

$$f(s,\sigma) \doteq \frac{RP}{1+a_0} \cdot \left(\sigma' + s' \cdot a_0 \cdot \frac{\sigma' - s' \cdot \frac{z_0 - \sigma'}{z_0}}{z_0} \right) \tag{9}$$

so that the inefficiency calculation is changed to

$$i_k \cdot RP_k = RP_k - \sum_{i=1}^{N_k} \hat{f}(s_i^{(k)}, \sigma_i^{(k)}, \bar{p}_i^{(k)}) \implies i_k = 1 - \sum_{i=1}^{N_k} \frac{\hat{f}(s_i^{(k)}, \sigma_i^{(k)}, \bar{p}_i^{(k)})}{RP_k}. \tag{10}$$

The actual pool rewards are now calculated after the inefficiency calculation, so that the total pool reward share changes to

$$R_k = (1 - \tau) \cdot (1 - i_k) \cdot (F_k + \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k). \tag{11}$$

The treasury mechanism is changed accordingly to

$$\begin{aligned} TR_{n+1} &= TR_n + \tau \cdot (1 - i_n) \cdot (F_n + \min\{\eta_n, 1\} \cdot \rho \cdot \Delta T_n) \\ &= TR_0 + \tau \cdot \sum_{k=0}^{n} (1 - i_k) \cdot (F_k + \min\{\eta_k, 1\} \cdot \rho \cdot \Delta T_k), \end{aligned} \tag{12}$$

which now includes only a contribute from the actually allocated part of the reward pot. The unallocated part of the reward pot goes back to the reserves

$$\Delta T_{k+1} = [1 - (1 - i_k) \cdot \min\{\eta_k, 1\} \cdot \rho] \cdot \Delta T_k + i_k \cdot F_k. \tag{13}$$

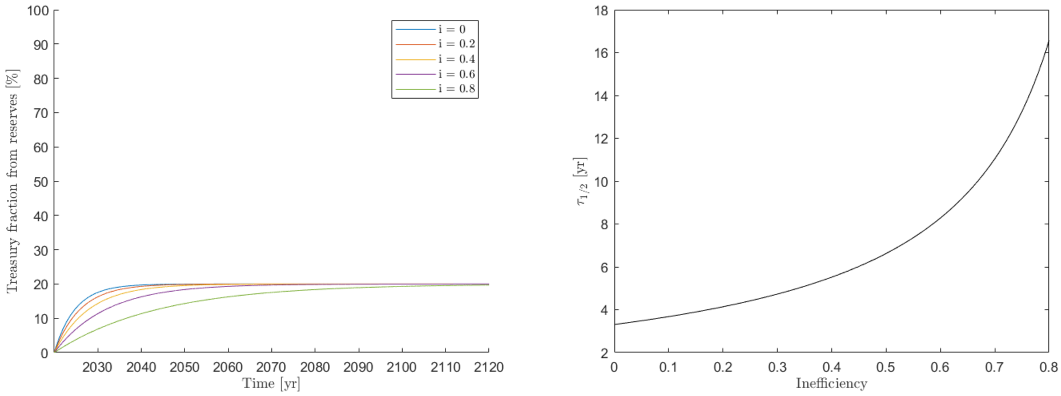
In order to simplify the analytical study of this mechanism, we employ the same assumptions as in the previous sections: $\forall k : F_k = 0$, $\eta_k = \eta < 1$ and $i_k = i$. The expected fraction of the reserves that went to the treasury at epoch n is now given by (see fig. 4 (a))

$$\frac{TR_n - TR_0}{\Delta T_0} = \tau \cdot (1 - e^{-n/\hat{\tau}}) \tag{14}$$

where we introduced the new effective time constant (see fig. 4 (b))

$$\hat{\tau}(\eta, i, \tau) = -\{\log[1 - (1 - i) \cdot \eta \cdot \rho]\}^{-1}. \tag{15}$$

With this new scheme the reserves allocated to the treasury are fixed for a given value of τ. Moreover, with this scheme the inefficiency affects more heavily the effective halving time, thus giving more time to the ecosystem to stabilize towards more efficiency.



(a) (b)

Figure 4: (a) Asymptotic behaviour of the expected fraction of the reserves that went to the treasury with CIP mechanism. (b) Effective half-life ($\tau_{1/2} = \ln 2 \cdot \hat{\tau}$) of the reserves as a function of the inefficiency i and network performance η with CIP mechanism. This is the time in years necessary to halve the reserves. $\eta = 0.956$ was estimated from actual network performance data $(0,956 \pm 0,035)$. The half-life diverges for $i = 1$, so the abscissa is cut close to $i = 0,8$ for a better comparison with figure 2 (b). $\eta = 0.956$ was estimated from actual network performance data $(0,956 \pm 0,035)$.

Acknowledgments

Our acknowledgments go to Lars Brünjes for his suggestions and for the exhortation to propose our solution through CIP.

References

[1] Philipp Kant, Lars Brünjes, and Duncan Coutts. Design Specification for Delegation and Incentives in Cardano. page 63.